Models and Methods for Solution of Interconnected Problems of Environment and Climate

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Goal

- The main idea of the lecture is in presentation of a next-generation scheme for solution of the problems of atmospheric physics and environment protection in which variational principles and methods of direct and inverse modeling with data assimilation are used.

Challenges of environment forecasting:

- **Predictability of climate-environment system?**

- **Stability of climatic system?**

- **sensitivity to perturbations of forcing**

Features of environment forecasting: uncertainty
- **in the long-term behavior of the climatic system;**
- **in the character of influence of man-made factors in the conditions of changing climate**

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Uncertainty

- **Discrepancy between models and real phenomena**

- **insufficient accuracy of numerical schemes and algorithms**

- **lack and errors of input data**

"If my husband would ever meet a woman on the street who looked like the women in his paintings, he would fall over in a dead faint." Mrs.Picasso

CONCEPT
OF ENVIRONMENTAL MODELING

The methodology is based on:

- **control theory,**
- **sensitivity theory,**
- **risk and vulnerability theory,**
    - variational principles in weak formulation,
    - combined use of models and observed data,
    - forward and inverse modeling procedures,
    - methodology for description of links between regional and global processes (including climatic changes) by means of orthogonal decomposition of functional spaces for analysis of data bases and phase spaces of dynamical systems

Basic elements for concept implementation:

- models of processes
- data and models of measurements
- global and local adjoint problems
- constraints on parameters and state functions
- functionals: objective, quality, control, restrictions etc.
- sensitivity relations for target functionals and constraints
- feedback equations for inverse problems

Mathematical model of processes

$$L(\boldsymbol{\varphi}, \mathbf{Y}) \equiv \mathbf{B}\frac{\partial \boldsymbol{\varphi}}{\partial t} + G(\boldsymbol{\varphi}, \mathbf{Y}) - \mathbf{f} - \mathbf{r} = 0$$

$$\boldsymbol{\varphi}^0 = \boldsymbol{\varphi}_a^0 + \boldsymbol{\xi}, \qquad \mathbf{Y} = \mathbf{Y}_a + \boldsymbol{\zeta};$$

$\boldsymbol{\varphi} \in \Im(D_t)$ state function,

$\mathbf{Y} \in \Re(D_t)$ parameter vector.

G "space" operator of the model

Variational form

$$I(\boldsymbol{\varphi}, \mathbf{Y}, \boldsymbol{\varphi}^*) \equiv \int_{D_t} (L(\boldsymbol{\varphi}, \mathbf{Y}), \boldsymbol{\varphi}^*) dDdt = 0$$

- $\boldsymbol{\varphi}^* \in \Im^*(D_t)$ adjoint functions
- $\mathbf{r}$, $\boldsymbol{\xi}, \boldsymbol{\varsigma}$ **are the terms describing uncertainties and errors of the corresponding objects.**

Model of atmospheric dynamics

$$\frac{\partial u}{\partial t} + \mathbf{v}\cdot\nabla u - fv + kw = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_u$$

$$\frac{\partial v}{\partial t} + \mathbf{v}\cdot\nabla v + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_v$$

$$\frac{\partial w}{\partial t} + \mathbf{v}\cdot\nabla w - ku = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_w$$

$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p + (\frac{c_p}{c_v}\nabla\cdot\mathbf{v})p = \left(\frac{c_p}{c_v} - 1\right)\rho c_p Q_T$$

$$\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T + (\frac{R_d}{c_v}(1+\alpha)\nabla\cdot\mathbf{v})T = \frac{c_p}{c_v}Q_T$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\mathbf{v} = 0, \quad \rho = p\left(R_d(1+\alpha)T\right)^{-1}$$

Transport and transformation of humidity

$$\frac{\partial q_v}{\partial t} + \mathbf{v} \cdot \nabla q_v = -\left(S_l + S_f\right) + F_{q_v}$$

$$\frac{\partial q_c}{\partial t} + \mathbf{v} \cdot \nabla q_c = S_c + F_{q_c}$$

$$\frac{\partial q_l}{\partial t} + \mathbf{v} \cdot \nabla q_l + \frac{1}{\rho}\frac{\partial}{\partial z}\rho q_l \left|v_{lT}\right| = S_l + F_{q_l}$$

$$\frac{\partial q_f}{\partial t} + \mathbf{v} \cdot \nabla q_f + \frac{1}{\rho}\frac{\partial}{\partial z}\rho q_f \left|v_{fT}\right| = S_f + F_{q_f}$$

Transport and transformation model
of gas pollutants and aerosols

$$\left(\mathrm{L}\boldsymbol{\varphi}\right)_i \equiv \frac{\partial \boldsymbol{\pi}\boldsymbol{\varphi}_i}{\partial t} + \mathrm{div}\,\boldsymbol{\pi}(\boldsymbol{\varphi}_i\mathbf{u} - \boldsymbol{\mu}_i \,\mathrm{grad}\,\boldsymbol{\varphi}_i) + \boldsymbol{\pi}((\mathrm{S}\boldsymbol{\varphi})_i - \mathrm{f}_i(\mathrm{x},\mathrm{t}) - \mathrm{r}_i) = 0,$$

Operators of transformation

$$\left\{\mathrm{S}_g(\boldsymbol{\varphi}) = \mathrm{P}(\boldsymbol{\varphi}) - \boldsymbol{\Pi}(\boldsymbol{\varphi})\right\}_i \equiv \sum_{q=1}^{R_i} \left\{ \mathrm{k}(q)\left(\mathrm{s}_i(q^-) - \mathrm{s}_i(q^+)\right) \prod_{j=1}^{U_r} \boldsymbol{\varphi}_j^{\mathrm{s}_j(q^-)} \right\}$$

$$S_a\left(\varphi_i(\sigma)\right) = \frac{1}{2}\int_{\sigma_0}^{\sigma}\left[\sum_{k=1}^{M}\gamma_{ik}\varphi_k(\sigma_1)\left(\sum_{m=1}^{M}\alpha_{km}K(\sigma-\sigma_1,\sigma_1)\varphi_k(\sigma-\sigma_1)\right)\right]d\sigma_1$$

$$-\varphi_i(\sigma)\int_{\sigma_0}^{\sigma_M} K(\sigma,\sigma_1)\left(\sum_{k=1}^{M}\beta_{ik}\varphi_k(\sigma_1)\right)d\sigma_1 - \frac{\partial}{\partial\sigma}\left[e_i\varphi_i(\sigma)\right] + \frac{\partial^2}{\partial\sigma^2}\left[\nu_i\varphi_i(\sigma)\right]$$

$$-R_i\varphi_i(\sigma) + Q_i(\sigma), \quad i = \overline{1,M}$$

Variational form of transport and transformation models

$$I\left(\boldsymbol{\varphi}, \mathbf{Y}, \boldsymbol{\varphi}^*\right) \equiv \sum_{i=1}^{n} \left\{ \left(\boldsymbol{\Lambda}\boldsymbol{\varphi}, \boldsymbol{\varphi}^*\right)_i + \int_{D_t} \boldsymbol{\pi}((S\boldsymbol{\varphi})_i - f_i(x,t) - r_i)\boldsymbol{\varphi} dDdt \right\} = 0$$

$$\left(\boldsymbol{\Lambda}\boldsymbol{\varphi}, \boldsymbol{\varphi}^*\right)_i \equiv \left(\int_{D_t} \left\{ 0.5 \left[\left(\boldsymbol{\varphi}^* \frac{\partial \boldsymbol{\pi}\boldsymbol{\varphi}}{\partial t} - \boldsymbol{\varphi} \frac{\partial \boldsymbol{\pi}\boldsymbol{\varphi}^*}{\partial t} \right) + \left(\boldsymbol{\varphi}^* \operatorname{div} \boldsymbol{\pi}\boldsymbol{\varphi}\mathbf{u} - \boldsymbol{\varphi} \operatorname{div} \boldsymbol{\pi}\boldsymbol{\varphi}^*\mathbf{u} \right) \right] \right. \right.$$

$$\left. + \boldsymbol{\pi}\boldsymbol{\mu} \operatorname{grad} \boldsymbol{\varphi} \operatorname{grad} \boldsymbol{\varphi}^* \right\} dDdt + \int_D 0.5\ \boldsymbol{\varphi}\boldsymbol{\varphi}^* \boldsymbol{\pi} dD \Big|_0^{\bar{t}} +$$

$$\left. \int_{\boldsymbol{\Omega}_t} \left(0.5\ \boldsymbol{\varphi} u_n - \boldsymbol{\mu} \frac{\partial \boldsymbol{\varphi}}{\partial n} \right) \boldsymbol{\varphi}^* \boldsymbol{\pi} d\boldsymbol{\Omega} dt + \int_{\boldsymbol{\Omega}_t} \left(R_b \boldsymbol{\varphi} - q_b \right) \boldsymbol{\varphi}^* \boldsymbol{\pi} d\boldsymbol{\Omega} dt \right)_i$$

$R_b \boldsymbol{\varphi} - q_b = 0$ boundary conditions on $\boldsymbol{\Omega}_t$

Model of observations

- **A set of measured data** φ_m, Ψ_m **on** D_t^m

$$\Psi_m = [H(\varphi)]_m + \boldsymbol{\eta},$$

$[H(\varphi)]_m$ **models of observations**.

Variational form

$$I_m(\varphi, \mathbf{Y}, \varphi\ ) \equiv \int_{D_t} \left[\left(\Psi_m - [H(\varphi)]_m - \boldsymbol{\eta} \right)^T W_5 \boldsymbol{\eta}^* \right] \chi_m dDdt = 0$$

- $\boldsymbol{\eta}$ **the term describing uncertainty and errors**
- $\boldsymbol{\eta}^*$ **adjoint function with respect to image of observation**
- $\chi_m dDdt$ **Radon's or Dirac's measure** $\left\{ \Psi(D_t^m) \Rightarrow \Psi(D_t\ ) \right\}$

Functionals for generalized description of information links in the system

Goal functionals

$$\Phi_k(\varphi) = \int_{D_t} F_k(\varphi)\chi_k(\mathbf{x},t)\,dDdt \equiv \left(F_k, \chi_k\right), \quad k = 1,\ldots,K$$

F_k are evaluated functions (differentiable in generirilized sense, bounded, satisfying the Lipschitz's conditions),

$\chi_k dDdt$ are Radon's or Dirac's measures on D_t, $\chi_k \in \Im^*(D_t)$.

Quality functionals for data assimilation

$$\Phi_k(\varphi) = \int_{D_t} (\Psi - H(\varphi))_m^T M (\Psi - H(\varphi))_m \chi_m(\mathbf{x},t)\,dDdt,$$

$\chi_m dDdt$ Radon's or Dirac's measures $\left\{\mathbf{\Psi}(D_t^m) \Rightarrow \mathbf{\Psi}(D_t)\right\}$

Functionals for generalized description of information links in the system

“Measurement” functionals for receptors

$$\Phi(\varphi) = \sum_{k=1}^{K} \int_{D_t} \left[H(\varphi)\right]_{rk} \delta(\mathbf{x} - \mathbf{x}_{rk}) dD\,dt,$$

$\mathbf{x}_{rk} \in D_t^r$ receptors locations,

$\delta(\mathbf{x} - \mathbf{x}_{rk}) dDdt$ Dirac’s measure for $D_t^r \Rightarrow D_t$

Functionals for assessment of distributed restrictions

$\varphi(\mathbf{x}, t) \leq N, \qquad \vartheta_k(\varphi(\mathbf{x}, t) \leq 0$ distributive constraints

$$\mathbf{\Phi}_k(\varphi) = \int_{D_t} (\vartheta_k(\varphi) + |\vartheta_k(\varphi)|) \chi_k(\mathbf{x}, t) dDdt = 0$$

$\chi_k dDdt$ are Radon’s or Dirac’s measures for constraints on $D_{tc} \subset D_t$,
$\chi_k \in \Im^*(D_t)$.

Variational principle

Augmented functional for computational technology

$$\tilde{\Phi}_k^h(\boldsymbol{\varphi}, \mathbf{Y}, \boldsymbol{\varphi}_k^*, \mathbf{r}, \boldsymbol{\xi}, \boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\eta}^*) = \Phi_k^h(\boldsymbol{\varphi}) +$$

$$+ \left[I(\boldsymbol{\varphi}, \mathbf{Y}, \boldsymbol{\varphi}_k^*) + I_m(\boldsymbol{\varphi}, \boldsymbol{\eta}, \boldsymbol{\eta}^*) \right]_{D_t^h} +$$

$$+0.5 \left\{ (\boldsymbol{\eta}^T W_1 \boldsymbol{\eta})_{D_t^m} + (\mathbf{r}^T W_2 \mathbf{r})_{D_t^h} + (\boldsymbol{\xi}^T W_3 \boldsymbol{\xi})_{D^h} + (\boldsymbol{\zeta}^T W_4 \boldsymbol{\zeta})_{R^h(D_t^h)} \right\}^h$$

Algorithms for construction of numerical schemes

$$\partial \tilde{\Phi}_k^h(\,,\ldots\,,)/\partial s = 0 \quad (s = \boldsymbol{\varphi}^*, \boldsymbol{\varphi}, \mathbf{Y}, \mathbf{r}, \boldsymbol{\xi}, \boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\eta}^*)^h$$

The universal algorithm of forward & inverse modeling

$$\frac{\partial \tilde{\Phi}_k^h}{\partial \boldsymbol{\varphi}^*} \equiv B\Lambda_t \boldsymbol{\varphi} + G^h(\boldsymbol{\varphi}, \mathbf{Y}) - \mathbf{f} - \mathbf{r} = 0$$

$$\frac{\partial \tilde{\Phi}_k^h}{\partial \boldsymbol{\varphi}} \equiv (B\Lambda_t)^T \boldsymbol{\varphi}_k^* + A^T(\boldsymbol{\varphi}, \mathbf{Y})\boldsymbol{\varphi}_k^* + \mathbf{d}_k = 0, \qquad \varphi_k^*(\mathbf{x})\Big|_{t=\bar{t}} = 0$$

$$d_k = \frac{\partial}{\partial \boldsymbol{\varphi}}\left(\boldsymbol{\Phi}_k^h(\boldsymbol{\varphi}) + \left(\left[(H(\boldsymbol{\varphi})\right]_m^T M_5 \boldsymbol{\eta}^*\right)\right), \qquad \boldsymbol{\varphi}^0 = \boldsymbol{\varphi}_a^0 + M_3^{-1}\boldsymbol{\varphi}_k^*(\mathbf{x}, 0), \quad t = 0$$

$$\mathbf{r}(\mathbf{x}, t) = M_2^{-1}\boldsymbol{\varphi}_k^*(\mathbf{x}, t), \qquad \mathbf{Y} = \mathbf{Y}_a - M_4^{-1}\boldsymbol{\Gamma}_k$$

$$\boldsymbol{\Gamma}_k = \frac{\partial}{\partial \mathbf{Y}} I^h(\boldsymbol{\varphi}, \mathbf{Y}, \boldsymbol{\varphi}_k^*)$$

$$A(\boldsymbol{\varphi}, \mathbf{Y})\delta\boldsymbol{\varphi} \equiv \frac{\partial}{\partial \alpha}\left[G^h(\boldsymbol{\varphi} + \alpha\delta\boldsymbol{\varphi}, \mathbf{Y})\right]_{\alpha=0}$$

$\boldsymbol{\Lambda}_t \boldsymbol{\varphi}$ is the approximation of time derivatives

Initial guess:

$$\mathbf{r}^{(0)} = 0, \quad \boldsymbol{\varphi}^{0(0)} = \boldsymbol{\varphi}_a^0, \quad \mathbf{Y}^{(0)} = \mathbf{Y}_a$$

Some elements
of optimal forecasting and design

The main sensitivity relations

Algorithm for calculation
of sensitivity functions

$$\delta\Phi_k^h(\varphi) \equiv (\Gamma_k, \delta\mathrm{Y}) \equiv \frac{\partial}{\partial\alpha} I^h(\varphi, \mathrm{Y} + \alpha\ \delta\mathrm{Y},\ \varphi_k^*)\Big|_{\alpha=0}$$

$$\Gamma_k = \frac{\partial}{\partial\delta\mathrm{Y}}\left(\frac{\partial}{\partial\alpha} I^h(\varphi, \mathrm{Y} + \alpha\ \delta\mathrm{Y}, \varphi_k^*)\Big|_{\alpha=0}\right)$$

Algorithms for uncertainty calculation based on sensitivity analysis and data assimilation:

in models of processes

$$\mathbf{r}(\mathbf{x},t) = M_2^{-1}\varphi_k^*(\mathbf{x},t),$$

in initial state

$$\xi = M_3^{-1} \quad \varphi_k^*(\mathbf{x},0), \qquad t = 0$$

in model parameters and sources

$$\vec{\zeta} = M_4^{-1}\vec{\Gamma}_k = M_4^{-1}\frac{\partial}{\partial \vec{Y}} I^h(\vec{\varphi},\vec{Y},\vec{\varphi}_k^*)$$

in models of observations

$$\eta^*(\mathbf{x},t) = M_5^{-1} M_1 \eta(\mathbf{x},t),$$

$M_i, (i=\overline{1,5})$ are weight matrices

Fundamental role of uncertainty functions

- integration of all technology components
- bringing control into the system
- regularization of inverse methods
- targeting of adaptive monitoring
- cost effective data assimilation

Optimal forecasting and design

Optimality is meant in the sense that estimations of the goal functionals do not depend on the variations :

- of the sought functions in the phase spaces of the dynamics of the physical system under study
- of the solutions of corresponding adjoint problems that generated by variational principles
- of the uncertainty functions of different kinds which explicitly included into the extended functionals

Construction of numerical approximations

- variational principle
- integral identity
- splitting and decomposition methods
- finite volumes method
- local adjoint problems
- analytical solutions
- integrating factors

Idea and basic approximations

Differential operators of common kind in the models

$$0 = \int_{x_{i-1}}^{x_i} \left(L\varphi - f\right)\varphi^* dx = \int_{x_{i-1}}^{x_i} L^*\varphi^*\varphi dx +$$

$$\left(A\varphi, \varphi^*\right)\Big|_{x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i} f(x)\varphi^*(x)dx = 0$$

If $L^*\varphi^* = 0,$ then

$$\left(A\varphi, \varphi^*\right)\Big|_{x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i} f(x)\varphi^*(x)dx = 0, \quad i = \overline{2, n_x}$$

$\varphi^{*(\alpha)}(x), \; x_{i-1} \le x \le x_i, \; \alpha = 1,2$ integrating multipliers

Fundamental **analytical** solutions of local adjoint problems

$$\left\{\varphi_i^{*(1)} = 1, \quad \varphi_{i+1}^{*(1)} = 0\right\}, \quad \left\{\varphi_i^{*(2)} = 0, \quad \varphi_{i+1}^{*(2)} = 1\right\}, \; i = \overline{1, n_x - 1}$$

Application of local adjoint problems for convection-diffusion-reaction equation

$$L\varphi \equiv \frac{\partial \varphi}{\partial t} + u\frac{\partial \varphi}{\partial t} - \mu \frac{\partial^2 \varphi}{\partial x^2} + d\varphi = f$$

$$\int_{x_{i-1}}^{x_i} \left(\frac{\varphi^{j+1} - \varphi^j}{\Delta t} + \left(u\frac{\partial \varphi}{\partial t} - \mu \frac{\partial^2 \varphi}{\partial x^2} + d\varphi - f \right)^{j+1} \right) \varphi^* dx =$$

$$= \left[u\varphi\varphi^* - \left(\mu\varphi^* \frac{\partial \varphi}{\partial x} - \mu\varphi \frac{\partial \varphi^*}{\partial x} \right) \right]_{x_{i-1}}^{x_i} + \int_{x_{i-1}}^{x_i} \left(\tilde{d}\varphi^* - u\frac{\partial \varphi^*}{\partial x} - \mu \frac{\partial^2 \varphi^*}{\partial x^2} \right) \varphi dx$$

$$- \int_{x_{i-1}}^{x_i} \left(\frac{\varphi^j}{\Delta t} + f \right) \varphi^* dx = 0 \quad \tilde{d} = d + \frac{1}{\Delta t}$$

Discrete-analytical system of equations

$$L^*\varphi^* \equiv \tilde{d}\varphi^* - u\frac{\partial\varphi^*}{\partial x} - \mu\frac{\partial^2\varphi^*}{\partial x^2} = 0, \qquad x_{i-1} \le x \le x_i$$

$$\left\{\varphi_i^{*(1)} = 1, \quad \varphi_{i+1}^{*(1)} = 0\right\}, \quad \left\{\varphi_i^{*(2)} = 0, \quad \varphi_{i+1}^{*(2)} = 1\right\}, \; i = \overline{1, n_x - 1}$$

$$-\left(\mu_{i-1/2}\frac{\partial\varphi^{*(1)}}{\partial x}\varphi\right)_{i-1} + \left(\mu_{i+1/2}\frac{\partial\varphi^{*(2)}}{\partial x}\varphi\right)_{i+1} +$$

$$\left[\left(\mu_{i-1/2}\frac{\partial\varphi^{*(1)}}{\partial x} + u_{i-1/2}\varphi^{*(1)}\right)_{i-0} - \left(\mu_{i+1/2}\frac{\partial\varphi^{*(2)}}{\partial x} + u_{i+1/2}\varphi^{*(2)}\right)_{i+0}\right]\varphi_i =$$

$$-\int_{x_{i-1}}^{x_i}\left(\frac{\varphi^j}{\Delta t} + f\right)\varphi^{*(1)}(x)dx + \int_{x_i}^{x_{i+1}}\left(\frac{\varphi^j}{\Delta t} + f\right)\varphi^{*(2)}(x)dx$$

$$i = \overline{2, n_x - 1} \qquad R_{cell} = |u|\Delta x / \mu < 10$$

Variational principle and decomposition for parallel schemes

$$\sum_{j=2}^{J}\left\{\sum_{\alpha=1}^{r}\left\{\left[\left(\psi_{\alpha}^{j}-\varphi^{j-1}\right)+\Delta\tau_{\alpha j}\left(L_{\alpha}^{j}\psi_{\alpha}^{j}-f_{\alpha}^{j}\right)\right]\psi_{\alpha}^{*j}\right.\right.$$

$$\left.+\left[\psi_{\alpha}^{j}-\left(\sigma_{\alpha}\varphi_{\alpha}^{j}+\left(1-\sigma_{\alpha}\right)\varphi_{\alpha}^{j-1}\right)\right]\varphi_{\alpha}^{*j}\delta t_{j}\right\}$$

$$\left.+\left[\varphi^{j}-\frac{1}{r}\sum_{\alpha=1}^{r}\varphi_{\alpha}^{j}\right]\varphi^{*j}\delta t_{j}\right\}+\sum_{j=2}^{J}\sum_{\alpha=1}^{r}\Phi_{k\alpha}^{j}(\vec{\varphi},\vec{Y})$$

$$\Phi_{k}^{h}=\sum_{j=2}^{J}\sum_{\alpha=1}^{r}\Phi_{k\alpha}^{j}(\vec{\varphi},\vec{Y})=\sum_{j=2}^{J}\sum_{\alpha=1}^{r}\left(\int_{D}F_{k\alpha}^{j}(\vec{\varphi},\vec{Y})\chi_{k\alpha}^{j}(\vec{x},t)\,dD\right)\delta t_{j}$$

$\varphi^{1}(\vec{x},t_{0})$ given

Scenario approach for environmental needs

- Inclusion of climatic data via decomposition of phase spaces on set of orthogonal subspaces ranged with respect to scales of perturbations
- Construction of deterministic and deterministic-stochastic scenarios on the basis of orthogonal subspaces
- Models with leading phase spaces

Methods of orthogonal decomposition of phase spaces of climate- environment information

- Revealing elements of long-term memory of the climatic system
- Analysis of variability of multi-component 4D spaces
- analysis of climate as realization of dynamic system behavior
- Analysis of climatic data in term of orthogonal subspaces gives the possibility to define dominant patterns in general system and to use them for scenario constructions

Method of orthogonal data decomposition

1. Data base is presented as $\left(N \times M\right)$-matrix

$$A = \left\{\boldsymbol{\varphi}_i = \left\{\varphi_i(k)\right\}\right\} \equiv a_{j,}^{(i)}, \quad i = \overline{1,M}, \quad j = \overline{1,N}, \quad M < N$$

i - column index, M - number of columns,
j - row index, N - number of rows.
k - set of multi-indeces of column-vectors.

2. Singular decomposition of $\left(N \times M\right)$-matrix A on a set of orthogonal subspaces of right and left singular vectors of matrix A, ranged with respect to magnitude of eigenvalues of $\left(M \times M\right)$- Gram matrix $A^* A$.

Data decomposition algorithm

$$A = V_N U_M,\ V_N \in R_N,\ U_M \in R_M \qquad (1)$$

$$A\mathbf{u} = \sigma\mathbf{v} \Rightarrow AA^*\mathbf{v} = \sigma^2\mathbf{v} \equiv \lambda\mathbf{v} \qquad (2)$$

$$A^*\mathbf{v} = \sigma\mathbf{u} \Rightarrow A^*A\mathbf{u} = \sigma^2\mathbf{u} \equiv \lambda\mathbf{u} \qquad (3)$$

1. Solution of the spectral problem (3)

$$\left\{\mathbf{u}_i; \lambda_i = \sigma_i^2\right\},\ \left(\mathbf{u}_i^T, \mathbf{u}_\alpha\right) = \delta_{i\alpha}\lambda_i,\ \ i,\alpha = \overline{1, M_0}$$

$$\left\{\lambda_1 \geq \lambda_2 \geq \mathrm{K} \geq \lambda_{M_0} > 0, \quad M_0 \leq M\right\},\ U_M = \left\{\mathbf{u}_i,\ i = \overline{1, M_0}\right\}$$

2. Calculation of the left singular vectors

$$V_N = \Sigma_{M_0}^{-1} A U_M, \quad \Sigma_{M_0} = \mathrm{diag}\left\{\sigma_i,\ i = \overline{1, M_0}\right\}$$

$$\left(\mathbf{v}_i^T, \mathbf{v}_\alpha\right) = \delta_{i\alpha},\ \ i,\alpha = \overline{1, M_0}, \quad V_N = \left\{\mathbf{v}_i,\ i = \overline{1, M_0}\right\}$$

3. Calculation of vectors for presentation (1)

$$\varphi_i(k) = \sum_{p=1}^{M_0} v_p(k) u_p(i) \quad i = \overline{1, M_0},\ \ k = \overline{1, M}$$

Conclusion

- Algorithms for optimal environmental forecasting and design are proposed
- The fundamental role of uncertainty is highlighted

Advantage of the approach

- Consistency of all technology elements
- Optimality of numerical schemes based on discrete-analytical approximations(without flux-correction procedures)
- Cost-effectiveness of computational technology

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